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THE CANONICAL-BOUNDARY REPRESENTATION FOR AUTOMORPHISM GROUPS OF LOCALLY COMPACT COUNTABLE STATE MARKOV SHIFTS

BY

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ABSTRACT

We introduce the canonical-boundary representation and study its range. This conjugacy invariant homomorphism captures information about the symmetry of the Markov shift near its (canonical) boundary and exhibits which actions on the boundary can be realized by automorphisms.

The path-structure at infinity — a relation on the set of orbits of the canonical boundary — is a new conjugacy invariant, which is stronger than the canonical boundary and the periodic data at infinity. Moreover we determine its influence on the range of the canonical-boundary representation and the extendability of automorphisms from subsystems (ascending sequences of shifts os finite type (SFTs) and infinite subsets of periodic points) to the entire Markov shift.

1. Preliminaries, outline and notations

We started to investigate the properties of the automorphism groups of topological countable state Markov shifts in [7] and [8]. Beside the direct methods used there one may focus on the study of the automorphism group via representations and group actions.

Just as in the SFT-case there are the well-known periodic-point and periodic-orbit representations mapping the group $\mathrm{Aut}(\sigma)$ of all shift commuting self-homeomorphisms into a direct product of countably many symmetric groups $\mathcal{S}_{\mathrm{Per}_n^0(\sigma)}$ or $\mathcal{S}_{\mathrm{Orb}_n(\sigma)}$ $(n \in \mathbb{N})$. These representations, first introduced by M. Boyle and W. Krieger [1], assign to any automorphism respectively its action

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on the set of periodic points or periodic orbits and are thus faithful (or almost faithful with kernel $\langle\sigma\rangle$) for subshifts with dense periodic points. The only difference for subshifts on a countably infinite alphabet is the possibility to have infinitely many periodic points/orbits for a given period and so the symmetric group on countably many objects may show up in the direct product.

As we have seen in [8] the automorphism groups of countable state Markov shifts have a very complicated structure, so instead of (almost) injective homomorphisms one would rather look at representations that suppress part of this complexity but still yield useful information about the subshift. To this end we define a new representation of $\mathrm{Aut}(\sigma)$ for locally compact countable state Markov shifts which records information about the structure of the subshift near its boundary, thus this may be regarded a line of vision from outside the system to some extent.

Since every automorphism of some locally compact countable state Markov shift extends onto the canonical compactification — its explicit construction is reviewed in Section 2 — originally defined by D. Fiebig and U.-R. Fiebig in [3], thinking of the canonical boundary as a compact-metric dynamical system in its own right, yields a homomorphism from the automorphism group of the Markov shift into the much smaller and simpler automorphism group of this boundary system. We coined the term canonical-boundary representation for this new invariant of topological conjugacy.

In Section 3 we show that this representation reflects the operation of any automorphism near the boundary. Its kernel contains all automorphisms acting on points via the modification of symbols from some finite subset of the alphabet only (i.e. all coordinates with symbols not inside this finite set stay unchanged). Its image can be stated explicitly in various examples. Non-surjectivity exhibits restrictions emanating from the subshift's structure at infinity: With increasing symmetry the image grows larger, whereas for thinned-out Markov shifts (and many other examples) it is forced to be minimal (Prop. 3.5), due to a lack of symmetry. Further restrictions on the image stem from the fact that any automorphism has to respect the period of its subshift (Theo. 3.6), hence two orbits of the canonical boundary cannot be rotated by a distinct amount modulo this period and the distances between pairs of preimages and corresponding image points have to coincide modulo the subshift's period. Moreover we prove the canonical-boundary representation to be an invariant much finer than the canonical boundary or the periodic data at infinity (see Section 3 for exact definitions).

In Section 4 we introduce a binary relation on the orbits of the canonical boundary, at first given in terms of a graph presentation. It describes the path-structure near any of the boundary-orbits and can be interpreted as a refinement of transition entropy. We prove independence of the graph presentation, so this relation yields a new invariant and we state some condition necessary for the existence of a topological conjugacy between two locally compact countable state Markov shifts in terms of this path-structure relation (Theo. 4.2). Since automorphisms are self-conjugacies from a subshift onto itself, we can use this condition to further restrict the image of the canonical-boundary representation. Rounding off the paper we present two model applications of our results: Example 4.4 shows that not every permutation — compatible with the action of the shift and with some necessary conditions concerning convergence of periodic points to boundary points — on an infinite set of periodic points of fixed but arbitrary period length can be realized as the restriction of some automorphism of the locally compact Markov shift. Therefore the general LIFT-hypothesis for periodic points, asking whether all actions on subsets of periodic points that satisfy all obvious constraints can be extended to automorphisms of the whole subshift (see [2] Section 7 and [4]), does not hold. As a second application (Ex. 4.5) we disprove the corresponding LIFT-hypothesis for sequences of embedded compact subshifts (even for SFTs), i.e. we show that in general a projective sequence of automorphisms, acting on a nested sequence of embedded compact subshifts/SFTs whose union is dense inside the locally compact Markov shift under consideration, cannot be extended to an automorphism of the whole Markov shift, even if the automorphisms in this sequence do not violate any conditions posed by the canonical boundary.

We give a short introduction to the notions used in this paper. For a general overview in symbolic dynamics have a look at the books by B. Kitchens [5] or by D. Lind and B. Marcus [6].

Let $\mathcal{A}^{\mathbb{Z}}$ denote the set of bi-infinite sequences over a countably infinite alphabet $\mathcal{A}$. $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology of the discrete topology on $\mathcal{A}$ is a non-compact, totally disconnected, perfect metric space. The shift map σ: $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $\sigma((x_i)_{i\in\mathbb{Z}}) := (x_{i+1})_{i\in\mathbb{Z}}$ is a homeomorphism inducing the dynamics on $\mathcal{A}^{\mathbb{Z}}$.

Any shift-invariant subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ endowed with the subspace topology generated by the countable set of clopen cylinders

$$_n[a_0 \dots a_m] := \{(x_i)_{i\in\mathbb{Z}} \in X \mid \forall 0 \le i \le m: \ x_{n+i} = a_i\} \quad (n \in \mathbb{Z},\ m \in \mathbb{N}_0)$$

yields a subshift (X, σ).

Two subshifts (X_1, σ_1), (X_2, σ_2) are (topologically) conjugate, if there exists a shift commuting homeomorphism γ: $X_1 \to X_2$. Pres(X) denotes the set of presentations of the subshift (X, σ), i.e. the set of all subshifts conjugate to (X, σ).

A subshift (X, σ) is called a countable state Markov shift, if its set of presentations contains an edge shift (X_G, σ) on some directed graph $G = (V, E)$ with $|E| = \aleph_0$ and σ acting on the set

$$X_G := \left\{(x_i)_{i\in\mathbb{Z}} \in E^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : \ \mathfrak{t}(x_i) = \mathfrak{i}(x_{i+1})\right\}$$

of bi-infinite walks along the edges of G. Here $\mathfrak{i}, \mathfrak{t}$: $E \to V$ are the initial and terminal vertex maps on the edge set E.

A countable state Markov shift (X, σ) is locally compact, if and only if X is locally compact, if and only if any (every) cylinder set is compact, if and only if for any (every) graph presentation (X_G, σ) the graph G is locally finite. Furthermore, (X, σ) is (topologically) transitive, if and only if X is irreducible, if and only if G is strongly connected.

A map φ: $X \to X$ is called an automorphism of (X, σ), if φ is a shift commuting homeomorphism from X onto itself. Obviously the set of automorphisms forms a group Aut(σ), which is a conjugacy invariant reflecting the inner structure and symmetries of the subshift.

2. Short review of the canonical compactification

We briefly recall the construction of the canonical compactification for locally compact Markov shifts given in [3]. It is the unique maximal element within the set of all metric compactifications whose boundary is an inverse limit of finite dynamical systems ([3], Theo. B(a)). The boundary is a conjugacy invariant ([3], Cor. 3.6(b)) capturing a good deal of the properties of the Markov shift 'at infinity' and can be built up from any graph presentation in the following way:

Let (X, σ) be some transitive, locally compact countable state Markov shift presented as an edge shift on some locally finite directed graph $G = (V, E)$. After removal of a finite set $K \subsetneq E$ of edges the remaining graph $G - K := (V, E \setminus K)$ decomposes into a finite set of infinite, maximal connected subgraphs called components $\mathcal{C}(G - K)$ and a finite (possibly not connected) remnant, which is not considered in the following. In contrast to the strong connectedness of G the components are assumed merely connected, i.e. any pair of vertices from one component can be connected via a path in the undirected graph underlying that component.

For any component $H = (V_H, E_H) \in \mathcal{C}(G - K)$ we define the weighted period $\mathrm{per}_w(H)$ as the greatest common divisor of the weights of all loops in the undirected graph underlying H. The weight of such an undirected loop $e_1, e_2, \ldots, e_n \in E_H$; $v_1, v_2, \ldots, v_n \in V_H$ with $\{\mathfrak{i}(e_k), \mathfrak{t}(e_k)\} = \{v_k, v_{k+1}\}$ for $k \in \{1, 2, \ldots, n-1\}$ and $\{\mathfrak{i}(e_n), \mathfrak{t}(e_n)\} = \{v_n, v_1\}$ is the sum of all edge-weights w_k, where $w_k := 1$ if and only if $\mathfrak{i}(e_k) = v_k$ and $w_k := -1$ otherwise ($k \in \{1, 2, \ldots, n\}$).

We point out that for general directed graphs G the weighted period $\mathrm{per}_w(G)$ may differ from the period $\mathrm{per}(G)$ defined as the greatest common divisor of the lengths of all directed loops in G ($\mathrm{per}_w(G)$ can be a proper divisor of $\mathrm{per}(G)$). However both notions coincide for strongly connected graphs — see Lemma 2.1. Thus as our subshifts are assumed transitive, we do not have to distinguish between the weighted period and the usual period of the whole graph (see Theo. 3.6); but we do have to distinguish between these notions if considered for components.

LEMMA 2.1: *For strongly connected graphs $G = (V, E)$ the weighted period $\mathrm{per}_w(G)$ equals the usual period*

$$\mathrm{per}(G) := \gcd\{n \in \mathbb{N} \mid \exists \text{ directed loop of length } n \text{ in } G\}.$$

Proof: Denote the set of all weights appearing between $u \in V$ and $v \in V$ with $W_G(u, v) := \{w \in \mathbb{Z} \mid \exists \text{ undirected path of weight } w \text{ connecting } u \text{ and } v\}$. As G is connected, $W_G(v, v)$ is independent of $v \in V$ and $\mathrm{per}_w(G)$ is a generator for this ideal, i.e. $W_G(v, v) = \mathrm{per}_w(G) \cdot \mathbb{Z}$.

Every directed loop at v of length $n \in \mathbb{N}$ has weight n, so $n \in W_G(v, v)$. Running through such a loop in reversed direction yields $-n \in W_G(v, v)$; concatenations finally show $n \cdot \mathbb{Z} \subseteq W_G(v, v)$. So $\gcd\{n \in \mathbb{N} \mid \exists$ directed loop of length n in $G\} \cdot \mathbb{Z} \subseteq \mathrm{per}_w(G) \cdot \mathbb{Z}$, which is equivalent to $\mathrm{per}_w(G) | \mathrm{per}(G)$.

For the opposite use that G is strongly connected: Take an undirected, closed path p of weight $\mathrm{per}_w(G)$. p can be divided into directed subpaths $p_1, p_2, \ldots, p_n$ ($n \in \mathbb{N}$) that are oriented alternatingly (w.l.o.g. let p_1 count positive), that is $\mathfrak{t}(p_1) = \mathfrak{t}(p_2)$; $\mathfrak{i}(p_2) = \mathfrak{i}(p_3)$; $\ldots$; $\mathfrak{f}(p_n) = \mathfrak{i}(p_1)$ with $\mathfrak{f}(p_n) := \begin{cases} \mathfrak{i}(p_n) & \text{iff } n \equiv 0\ (2) \\ \mathfrak{t}(p_n) & \text{iff } n \equiv 1\ (2) \end{cases}$
Then the total weight is the alternating sum of their lengths:

$$\mathrm{per}_w(G) = |p_1| - |p_2| + |p_3| - \cdots + (-1)^{n-1} |p_n| .$$

For every subpath p_{2i} ($i \in \left\{1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$) choose a directed path p'_{2i} connecting $\mathfrak{i}(p'_{2i}) = \mathfrak{t}(p_{2i})$ to $\mathfrak{t}(p'_{2i}) = \mathfrak{i}(p_{2i})$. Obviously $\mathrm{per}(G) | (|p_{2i}| + |p'_{2i}|)$. Now $q := p_1 p'_2 p_3 p'_4 \cdots p_n^{(\prime)}$ is a directed loop, so $\mathrm{per}(G) \mid |q|$. The result follows from putting together all the relations:

$$\mathrm{per}(G) | (\underbrace{|p_1| + |p'_2| + |p_3| + \cdots + |p_n^{(\prime)}|}_{|q|}) - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (|p_{2i}| + |p'_{2i}|) = \mathrm{per}_w(G). \qquad \blacksquare$$

To construct the canonical boundary choose an ascending sequence $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of finite subsets in E, such that $\bigcup_{n \in \mathbb{N}} K_n = E$. This gives a sequence of finite dynamical systems (A_n, θ_n)

$$A_n := \{(H, i) | H \in \mathcal{C}(G - K_n) \wedge \ 0 \le i < \mathrm{per}_w(H)\}$$
$$\theta_n \colon A_n \to A_n \quad \theta_n((H, i)) := (H, (i + 1) \bmod \mathrm{per}_w(H))$$

containing for every component $H \in \mathcal{C}(G - K_n)$ one periodic θ_n-orbit of length $\mathrm{per}_w(H)$.

As the sequence $(K_n)_{n \in \mathbb{N}}$ of subsets is ascending one has for any component $H \in \mathcal{C}(G - K_{n+1})$ a unique component $H' \in \mathcal{C}(G - K_n)$ containing H. This yields projections $\pi_n \colon A_{n+1} \to A_n$, $\pi_n((H, i)) := (H', i \bmod \mathrm{per}_w(H'))$ gluing together all the (A_n, θ_n).

It is easy to check $\mathrm{per}_w(H), \mathrm{per}_w(H') > 0$ and $\mathrm{per}_w(H') | \mathrm{per}_w(H)$ (see Lemma 3.2 and the adjacent text in [3]), thus the commutation relation $\pi_n \circ \theta_{n+1} = \theta_n \circ \pi_n$ is valid for any $n \in \mathbb{N}$.

The inverse limit induced by the projections $(\pi_n)_{n \in \mathbb{N}}$ on $((A_n, \theta_n))_{n \in \mathbb{N}}$ gives a dynamical system $(Z(X, (K_n)_{n \in \mathbb{N}}), \Theta_X)$ — the canonical boundary of (X, σ):

$$Z(X, (K_n)_{n \in \mathbb{N}}) := \{((H_n, i_n) \in A_n)_{n \in \mathbb{N}} | \forall n \in \mathbb{N} : \pi_n((H_{n+1}, i_{n+1})) = (H_n, i_n)\}$$
$$\Theta_X \colon Z(X, (K_n)_{n \in \mathbb{N}}) \to Z(X, (K_n)_{n \in \mathbb{N}}), \ \Theta_X((H_n, i_n)_{n \in \mathbb{N}}) := (\theta_n(H_n, i_n))_{n \in \mathbb{N}}$$

Put on $\prod_{n \in \mathbb{N}} A_n$ the product topology of the discrete topologies on the finite sets A_n, then $Z(X, (K_n)_{n \in \mathbb{N}}) \subseteq \prod_{n \in \mathbb{N}} A_n$ equipped with the induced subspace topology becomes a compact-metric space and Θ_X is a homeomorphism. Lemma 3.4 in [3] shows that the dynamical system $(Z(X, (K_n)_{n \in \mathbb{N}}), \Theta_X)$ is in fact independent of the choice of the sequence $(K_n)_{n \in \mathbb{N}}$, so we use the shorthand $(Z(X), \Theta_X)$.

Attaching $(Z(X), \Theta_X)$ as a boundary to the transitive, locally compact countable state Markov shift (X, σ) we get the canonical compactification $(\hat{X}, \hat{\sigma})$ with $\hat{X} := X \dot{\cup} Z(X)$ and $\hat{\sigma} \colon \hat{X} \to \hat{X}$ such that $\hat{\sigma}|_X := \sigma$, $\hat{\sigma}|_{Z(X)} := \Theta_X$.

This gluing is done by means of an intrinsic factor map $g\colon X \to Z(X)$ defined as follows: Choose an edge $e \in E$, put $g_G(e) := 0$ and extend this to a periodic labeling $g_G\colon E \to \{0,1,\ldots,\mathrm{per}_w(G)-1\}$ via $g_G(f_1)+1 \equiv g_G(f_2) \bmod \mathrm{per}_w(G)$ for all $f_1, f_2 \in E$ with $\mathfrak{t}(f_1) = \mathfrak{i}(f_2)$. For every component $H \in \mathcal{C}(G-K_n)$ $(n \in \mathbb{N})$ there exists a maximal-periodic labeling $g_H\colon E_H \to \{0,1,\ldots,\mathrm{per}_w(H)-1\}$ such that $g_H(e) \equiv g_G(e) \bmod \mathrm{per}_w(G)$ for all $e \in E_H$ and in addition these labelings form a projective family, i.e., respect the subgraph relation:

$$\forall H \in \mathcal{C}(G-K_{n+1}), H' \in \mathcal{C}(G-K_n)\colon H \subseteq H' \\ \Longrightarrow \forall e \in E_H : g_H(e) \equiv g_{H'}(e) \bmod \mathrm{per}_w(H')$$

This yields a locally constant map g; for all $x \in X$ with $x_0 \in K_1$ put $g(x) := z^*$ for an arbitrary $z^* \in Z(X,(K_n)_{n\in\mathbb{N}})$. Otherwise let $n := \max\{i \in \mathbb{N} \mid x_0 \notin K_i\}$ and $H \in \mathcal{C}(G-K_n)$ the unique component containing $x_0 \in E_H$ and set $g(x) := z$ for some $z \in Z(X,(K_n)_{n\in\mathbb{N}})$ with $z_n = (H, g_H(x_0))$. For the details in this construction the reader is referred to the original paper [3].

Finally we remark that the topology on $(\hat{X},\hat{\sigma})$ is given by an — up to uniform equivalence — unique (canonical) metric $\hat{d}\colon \hat{X}\times\hat{X} \to \mathbb{R}$, which according to [3] can be presented as:

$$\hat{d}(x,y) := d_0(\pi(x),\pi(y)) + \rho(\hat{g}(x),\hat{g}(y)) \quad \forall x,y \in \hat{X}$$

Here $d_0\colon X_0 \times X_0 \to \mathbb{R}$ denotes the Gurevich metric on the 1-point-compactification (X_0,σ_0); $\pi\colon \hat{X} \to X_0$, $\pi|_X := \mathrm{Id}_X$, $\pi(Z(X)) := \{\infty\}$ is the projection of the canonical onto the 1-point-compactification collapsing the canonical boundary $Z(X)$ down to the point ∞; $\rho\colon Z(X)\times Z(X) \to \mathbb{R}$ denotes any metric compatible with the topology on $(Z(X),\Theta_X)$ and $\hat{g}\colon \hat{X} \to Z(X)$, $\hat{g}|_X := g$, $\hat{g}|_{Z(X)} := \mathrm{Id}_{Z(X)}$ is the extension of the intrinsic factor map g.

3. The canonical-boundary representation

As we have seen in the previous section the canonical compactification and hence the canonical boundary are invariants. One of the main results of [3] (Theo. B(b)) states that every topological conjugacy between two transitive, locally compact countable state Markov shifts is even uniformly continuous with respect to the corresponding metrics $\hat{d}$ as defined above. Therefore each can be extended in exactly one way to a conjugacy between the canonical compactifications. Obviously this implies the unique extendability of any automorphism onto the canonical compactification, i.e. there is a canonical embedding

ε: $\mathrm{Aut}(\sigma) \hookrightarrow \mathrm{Aut}(\hat{\sigma})$, $\varphi \mapsto \hat{\varphi}$ with $\hat{\varphi}|_X = \varphi$ from the automorphism group of the Markov shift (X, σ) into that of its canonical compactification $(\hat{X}, \hat{\sigma})$.

Since any automorphism in $\varepsilon(\mathrm{Aut}(\sigma))$ leaves $X \subsetneq \hat{X}$ invariant, we may restrict its action to the complement $Z(X) = \hat{X} \setminus X$, to get an automorphism of the boundary system $(Z(X), \Theta_X)$. Composition of ε with this restriction map π: $\varepsilon(\mathrm{Aut}(\sigma)) \to \mathrm{Aut}(\Theta_X)$, $\hat{\varphi} \mapsto \hat{\varphi}|_{Z(X)}$ yields a homomorphism $\beta := \pi \circ \varepsilon$ from $\mathrm{Aut}(\sigma)$ into the smaller and often simpler automorphism group $\mathrm{Aut}(\Theta_X)$ of the canonical boundary interpreted as a compact-metric dynamical system. One easily checks that $\beta(\varphi \circ \phi) = \beta(\varphi) \circ \beta(\phi)$ and $\beta(\mathrm{Id}_X) = \mathrm{Id}_{Z(X)}$. This gives the desired representation of $\mathrm{Aut}(\sigma)$:

Definition 3.1: Let (X, σ) be a transitive, locally compact countable state Markov shift. The map β: $\mathrm{Aut}(\sigma) \to \mathrm{Aut}(\Theta_X)$, $\varphi \mapsto \hat{\varphi}|_{Z(X)}$ with $\hat{\varphi} \in \mathrm{Aut}(\hat{\sigma})$ the unique extension of φ onto the canonical compactification is called **canonical-boundary representation** of $\mathrm{Aut}(\sigma)$. Moreover β is a well-defined, conjugacy invariant group homomorphism.

Remark 3.2: Since the canonical compactification is the unique maximal element in the class of all metric compactifications having a boundary which is an inverse limit of finite dynamical systems (see [3], Theorem B(a)) the canonical-boundary representation resolves the inner structure of the Markov shift best possible. All other boundary representations would be coarser. This can be illustrated by the other 'extreme' compactification: Due to uniform continuity with respect to the Gurevich metric d_0 any automorphism of a locally compact countable state Markov shift (X, σ) can be extended to its 1-point-compactification (X_0, σ_0), but the corresponding homomorphism β_0: $\mathrm{Aut}(\sigma) \to \mathrm{Aut}(\mathrm{Id}_\infty) = \{\mathrm{Id}_\infty\}$ would be trivial and thus does not contain any useful information about the structure of (X, σ) near the boundary.

Now we start to investigate the properties of β. Obviously this homomorphism is not faithful; since many locally compact countable state Markov shifts have canonical boundaries consisting of finitely many periodic Θ_X-orbits, $\mathrm{Aut}(\Theta_X)$ may be finite, whereas $\mathrm{Aut}(\sigma)$ is at least countably infinite ([8] Theo. 2.4). So in general β has to collapse automorphisms. Another fact immediately destroying injectivity is that the kernel of the canonical-boundary representation contains all automorphisms $\varphi \in \mathrm{Aut}(\sigma)$ acting just on a finite subset of symbols, i.e. it exists $F \subsetneq \mathcal{A}$ finite, such that for all $x \notin \bigcup_{f \in F} {}_0[f]$ the zero-coordinate of x stays unchanged under the action of φ:

Proposition 3.3: *Given a locally compact countable state Markov shift (X,σ), every automorphism acting exclusively on a finite subset of symbols from the alphabet $\mathcal{A}$ induces the identity on the canonical boundary. Therefore all those automorphisms are elements of* $\ker(\beta)$.

Proof: Let $F \subsetneq \mathcal{A}$ finite such that $(\varphi(x))_0 = x_0$ for all $x \in X$ with $x_0 \notin F$ and denote by $(x^{(n)})_{n\in\mathbb{N}}$ a sequence of points $x^{(n)} \in X$ converging with respect to the $\hat{d}$-metric to some boundary point $z \in Z(X)$. Thus

$$\begin{aligned} 0 = \lim_{n\to\infty} \hat{d}(x^{(n)}, z) &= \lim_{n\to\infty} d_0(\pi(x^{(n)}), \pi(z)) + \rho(\hat{g}(x^{(n)}), \hat{g}(z)) \\ &= \lim_{n\to\infty} d_0(x^{(n)}, \infty) + \rho(g(x^{(n)}), z) \end{aligned}$$

implies $\lim_{n\to\infty} d_0(x^{(n)}, \infty) = 0$, i.e., for n big enough larger and larger blocks $x^{(n)}_{[-j_n,j_n]}$ ($(j_n)_{n\in\mathbb{N}}$ an eventually increasing sequence of natural numbers) avoid F. Using the assumption for $|i| \le j_n$ yields:

$$(\varphi(x^{(n)}))_i = (\varphi(\sigma^i(x^{(n)})))_0 = x_i^{(n)} \notin F.$$

The triangle-inequality for $\hat{d}$ and the continuity of $\hat{\varphi}$ used in the following inequality

$$\lim_{n\to\infty} \hat{d}(x^{(n)}, \hat{\varphi}(z)) \le \lim_{n\to\infty} \underbrace{\hat{d}(x^{(n)}, \hat{\varphi}(x^{(n)}))}_{\to 0} + \underbrace{\hat{d}(\hat{\varphi}(x^{(n)}), \hat{\varphi}(z))}_{\to 0} = 0$$

finally give convergence of $(x^{(n)})_{n\in\mathbb{N}}$ to $\hat{\varphi}(z)$ forcing $\hat{\varphi}|_{Z(X)} = \mathrm{Id}_{Z(X)}$. ∎

Examples of automorphisms satisfying Proposition 3.3 are finite compositions of 1-block-automorphisms that permute all occurrences of symbols corresponding to a multi-edge in the graph. (A multi-edge is a finite set of edges connecting a common initial to a common terminal vertex.)

Next we study the image $\beta(\mathrm{Aut}(\sigma)) \le \mathrm{Aut}(\Theta_X)$ of the canonical-boundary representation. This tells us, which actions on the boundary system can be realized by automorphisms of the Markov shift. The size of $\beta(\mathrm{Aut}(\sigma))$ is related to the inner symmetry of the subshift (X,σ) near its boundary. Surjectivity signals the absence of further restrictions, whereas a lack of surjectivity reflects some kind of rigidness inside the Markov shift that does not surface in the boundary.

Of course such restrictions can only exist if the canonical boundary comprises more than one Θ_X-orbit, since $\Theta_X = \hat{\sigma}|_{Z(X)}$ implies $\langle\Theta_X\rangle \le \beta(\mathrm{Aut}(\sigma))$ and

$\mathrm{Aut}(\Theta_X) = \langle\Theta_X\rangle$ for boundaries $(Z(X), \Theta_X)$ with just one Θ_X-orbit. The corresponding transitive, locally compact countable state Markov shifts are characterized by the property that removing some finite set of edges $K \subsetneq E$ from any (every) graph presentation $G = (V, E)$ yields just one infinite connected component in $G - K$, i.e., there is only one direction coming from or going to infinity in G. An example is the topological random walk on $\mathbb{N}$ (e.g. with stepsize $0, \pm 1$).

In general transitive, locally compact countable state Markov shifts with a canonical boundary comprising more than one Θ_X-orbit do put strong restrictions on the boundary-automorphisms. This prohibits β from being surjective. To show this look at the class of thinned-out Markov shifts first considered in [8]:

Definition 3.4: A transitive, locally compact countable state Markov shift is called **thinned-out**, if and only if it is conjugate to an edge shift on some (strongly connected, locally finite, directed) graph $G = (V, E)$ with $|E| = \aleph_0$ containing a vertex $v \in V$ such that the set $L_v := \{l_n \mid n \in \mathbb{N}\}$ of first-return loops at v satisfies

$$\text{(GC)} \qquad \forall M \in \mathbb{N}_0 \exists N \in \mathbb{N}\, \forall n \geq N : |l_{n+1}| - |l_n| > M.$$

All thinned-out Markov shifts have such a rigid structure that there are no symmetries left near the boundary. This shows up immediately in the small size of $\beta(\mathrm{Aut}(\sigma))$:

PROPOSITION 3.5: *For every thinned-out Markov shift (X, σ) the image of the canonical-boundary representation is minimal, i.e. $\beta(\mathrm{Aut}(\sigma)) = \langle\Theta_X\rangle \cong \mathbb{Z}$. In particular this result does not depend on the size or structure of the canonical boundary.*

Proof: Since β is a conjugacy invariant, assume (X, σ) to be presented as an edge shift on some thinned-out graph $G = (V, E)$ satisfying (GC) for $v \in V$. From Theo. 6.4 in [8] one knows that the automorphism groups of thinned-out Markov shifts split into a direct sum $\mathrm{Aut}(\sigma) = \langle\sigma\rangle \oplus H$, where H consists exactly of those automorphisms $\phi \in \mathrm{Aut}(\sigma)$ that act on the orbit-complement of some finite subset $K_\phi \subsetneq E$ of edges like the identity.

Denote by $F_\phi := \{e \in E \mid \exists k \in K_\phi, l \in L_v : k \in l \wedge e \in l\}$ the set of all edges from first-return loops at v, that are marked by the edges in K_ϕ. As only finitely many elements in L_v are marked by K_ϕ (see [8], Lemma 7.4), F_ϕ is finite.

To prove $H \leq \ker(\beta)$ take $\phi \in H$ and choose a sequence $(x^{(n)})_{n\in\mathbb{N}}$ of points $x^{(n)} \in X$ converging to an arbitrary boundary point $z \in Z(X)$ with respect to $\hat{d}$. This implies the convergence $\lim_{n\to\infty} d_0(x^{(n)}, \infty) = 0$ which forces eventually larger and larger blocks $x^{(n)}_{[-j_n,j_n]}$ ($(j_n \in \mathbb{N})_{n\in\mathbb{N}}$ eventually increasing) to avoid the set F_ϕ and hence all K_ϕ-marked first-return loops at v.

W.l.o.g. the sequence $(x^{(n)})_{n\in\mathbb{N}}$ may be replaced by some sequence $(y^{(n)} \in X)_{n\in\mathbb{N}}$ with $y^{(n)}_{[-j_n,j_n]} = x^{(n)}_{[-j_n,j_n]}$, still converging to z but completely avoiding the set K_ϕ (and thus also the finite subset of first-return loops at v being marked by K_ϕ) for large $n \in \mathbb{N}$. That is eventually all points $y^{(n)}$ are elements of the orbit-complement $\mathrm{Orb}(K_\phi)^{\complement} := X \setminus \bigcup_{n\in\mathbb{Z}} \sigma^n(\bigcup_{k\in K_\phi} {}_0[k])$. Now $\phi|_{\mathrm{Orb}(K_\phi)^{\complement}} = \mathrm{Id}_{\mathrm{Orb}(K_\phi)^{\complement}}$ implies $\phi(y^{(n)}) = y^{(n)}$ for $n \in \mathbb{N}$ large enough and continuity of the extension $\hat{\phi} \in \mathrm{Aut}(\hat{\sigma})$ gives:

$$\lim_{n\to\infty} \hat{d}(y^{(n)}, z) = 0 \quad \Longrightarrow \quad 0 = \lim_{n\to\infty} \hat{d}(\hat{\phi}(y^{(n)}), \hat{\phi}(z)) = \lim_{n\to\infty} \hat{d}(y^{(n)}, \hat{\phi}(z)).$$

Therefore, $\hat{\phi}(z) = z$ for all $z \in Z(X)$ and $\phi \in \ker(\beta)$.

The above mentioned direct sum decomposition yields a unique representation $\varphi = \sigma^i \circ \phi$ with $i \in \mathbb{Z}$ and $\phi \in H$ for every $\varphi \in \mathrm{Aut}(\sigma)$. As desired one gets $\beta(\varphi) = \beta(\sigma^i \circ \phi) = \beta(\sigma)^i \circ \mathrm{Id}_{Z(X)} = {\Theta_X}^i \in \langle\Theta_X\rangle$, i.e. $\beta(\mathrm{Aut}(\sigma)) \leq \langle\Theta_X\rangle$. To show the contrary, observe that every element ${\Theta_X}^i \in \langle\Theta_X\rangle$ has $\sigma^i \in \mathrm{Aut}(\sigma)$ as a preimage.

Finally $\langle\Theta_X\rangle \cong \mathbb{Z}$ for thinned-out Markov shifts, since, otherwise, the sequence of weighted periods $(\mathrm{per}_w(H_n))_{n\in\mathbb{N}}$ would be bounded for every boundary point $z = (H_n, i_n)_{n\in\mathbb{N}} \in Z(X)$ in contradiction to (GC). ∎

The following theorem shows that the extension of any automorphism onto the canonical compactification has to respect the period of the subshift. The action on the canonical boundary has to be uniform in the sense that all Θ_X-orbits are rotated by the same degree and distinct boundary points have the same distance (with respect to the maximal-periodic labelling) modulo the subshift's period as their image points. This further restricts the image of β.

Theorem 3.6: *Denote by $G = (V, E)$ some graph presentation of a transitive, locally compact countable state Markov shift (X, σ). Let the canonical boundary $(Z(X), \Theta_X)$ be defined via a sequence $(K_n)_{n\in\mathbb{N}}$ ($\forall n \in \mathbb{N} : K_n \subseteq K_{n+1} \subsetneq E$ finite; $\bigcup_{n\in\mathbb{N}} K_n = E$) as described in Section 2. For any four boundary points $z_j = (H^{(j)}_n, i^{(j)}_n)_{n\in\mathbb{N}} \in Z(X)$ ($j \in \{1, 2, 3, 4\}$) and any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ with $\hat{\varphi}(z_1) = z_3$ and $\hat{\varphi}(z_2) = z_4$ it follows that $i^{(3)}_1 - i^{(1)}_1 \equiv i^{(4)}_1 - i^{(2)}_1 \mod \mathrm{per}(G)$.*

Proof: W.l.o.g. let all K_n ($n \in \mathbb{N}$) be edge sets of strongly connected subgraphs of G and assume there is an edge $e \in K_1$ with $g_G(e) = 0$. Here $g\colon X \to Z(X)$ is the intrinsic factor map used to attach $Z(X)$ to X and $g_G\colon E \to \{0, 1, \ldots, \mathrm{per}(G) - 1\}$ denotes the corresponding maximal-periodic labeling.

As each edge subset K_n is strongly connected there exists some right-infinite ray $r := r_1 r_2 r_3 \cdots$ starting at $\mathfrak{t}(e)$ and running directly through the infinite components $H_n^{(1)}$, i.e., there is an increasing sequence $(s_n \in \mathbb{N})_{n \in \mathbb{N}}$ such that $r_i \in E_{H_n^{(1)}} \setminus E_{H_{n+1}^{(1)}}$ for all $s_n \le i < s_{n+1}$. Akin choose a left-infinite ray $w := \cdots w_{-3} w_{-2} w_{-1}$ which ends at $\mathfrak{i}(e)$ such that there exists some increasing sequence $(t_n \in \mathbb{N})_{n \in \mathbb{N}}$ with $w_{-i} \in E_{H_n^{(2)}} \setminus E_{H_{n+1}^{(2)}}$ for all $t_n \le i < t_{n+1}$.

Concatenating these with e defines a point

$$x := \cdots w_{-3} w_{-2} w_{-1} . e r_1 r_2 r_3 \cdots \in X.$$

Now for every $n \in \mathbb{N}$ there is an index $k \in \mathbb{N}$ such that $(\sigma^k(x))_0 \in E_{H_n^{(1)}}$ and $g_{H_n^{(1)}}((\sigma^k(x))_0) = i_n^{(1)}$. By construction of r the σ-forward orbit of x contains some sequence $(\sigma^{k_n}(x))_{n \in \mathbb{N}}$ ($k_n \in \mathbb{N}$ for all $n \in \mathbb{N}$) of points converging to z_1 with respect to $\hat{d}$. Moreover, eventually all elements of this sequence are elements in $\mathcal{O}^+(x) := \{\sigma^k(x) | k \in \mathbb{N} \wedge k \equiv i_1^{(1)} \bmod \mathrm{per}(G)\}$ because $g_{H_1^{(1)}}((\sigma^k(x))_0) = i_1^{(1)}$ is possible only for $k \equiv i_1^{(1)} \bmod \mathrm{per}(G)$. Using the same argument on w yields a sequence $(\sigma^{-j_n}(x))_{n \in \mathbb{N}}$ ($j_n \in \mathbb{N}$ for all $n \in \mathbb{N}$) which converges to z_2 and is eventually contained in $\mathcal{O}^-(x) := \{\sigma^{-j}(x) | j \in \mathbb{N} \wedge -j \equiv i_1^{(2)} \bmod \mathrm{per}(G)\}$. Due to the continuity of $\hat{\varphi}$ at $z_3 = \hat{\varphi}(z_1)$ and $z_4 = \hat{\varphi}(z_2)$ one gets convergence $(\varphi(\sigma^{k_n}(x)))_{n \in \mathbb{N}} \overset{n \to \infty}{\longrightarrow} z_3$ and $(\varphi(\sigma^{-j_n}(x)))_{n \in \mathbb{N}} \overset{n \to \infty}{\longrightarrow} z_4$.

$$\begin{aligned} \text{Therefore} \quad i_1^{(3)} &\equiv g_G((\varphi(\sigma^{k_n}(x)))_0) \equiv g_G((\varphi(x))_0) + k_n \quad \bmod \mathrm{per}(G) \\ \text{and} \quad i_1^{(4)} &\equiv g_G((\varphi(\sigma^{-j_n}(x)))_0) \equiv g_G((\varphi(x))_0) - j_n \quad \bmod \mathrm{per}(G) \end{aligned}$$

for all $n \in \mathbb{N}$ large enough. Increasing n until $\sigma^{k_n}(x) \in \mathcal{O}^+(x)$ and $\sigma^{-j_n}(x) \in \mathcal{O}^-(x)$ implies the congruence

$$i_1^{(3)} - i_1^{(1)} \equiv g_G((\varphi(x))_0) \equiv i_1^{(4)} - i_1^{(2)} \bmod \mathrm{per}(G). \qquad \blacksquare$$

COROLLARY 3.7: *Let $(Z(X), \Theta_X)$ denote the canonical boundary of some transitive, locally compact countable state Markov shift (X, σ). It is not possible for the extension of any automorphism to rotate two Θ_X-orbits by distinct amounts modulo the subshift's period, that is, all $\varphi \in \mathrm{Aut}(\sigma)$ and $z_1, z_2 \in Z(X)$ satisfy $(\hat{\varphi}(z_1) = \hat{\sigma}^{i_1}(z_1) \wedge \hat{\varphi}(z_2) = \hat{\sigma}^{i_2}(z_2)) \Rightarrow i_1 \equiv i_2 \bmod \mathrm{per}(X)$.*

Proof: The result follows directly from Theorem 3.6, since for $k := 1, 2$ one has $\hat{\sigma}^{i_k}(z_k) = (H_n^{(k)}, i_n^{(k)} + i_k \operatorname{mod} \operatorname{per}_w(H_n^{(k)}))_{n \in \mathbb{N}}$ and $\operatorname{per}(X) | \operatorname{per}_w(H_n^{(k)})$ for all $n \in \mathbb{N}$. ∎

Finally the actions induced on the canonical boundary by automorphisms of the Markov shift have to respect the periodic data at infinity.

Definition 3.8: For any point $z \in Z(X)$ in the canonical boundary $(Z(X), \Theta_X)$ of some transitive, locally compact countable state Markov shift (X, σ) denote by $\Omega(z) := \{k \in \mathbb{N} | \exists (x^{(n)} \in \operatorname{Per}_k^0(\sigma))_{n \in \mathbb{N}} : \lim_{n \to \infty} \hat{d}(x^{(n)}, z) = 0\}$ the **periodic data at infinity** of z.

It is easy to check that the periodic data at infinity is a conjugacy invariant, such that the Ω-sets of preimage and image points under a topological conjugacy are equal. Obviously all the Ω-sets of boundary points belonging to the same Θ_X-orbit coincide. Therefore instead of the periodic data at ∞ of a single boundary point we could speak of the periodic data at ∞ of the corresponding Θ_X-orbit.

Remark 3.9: We point out, that Definition 3.8 slightly differs from the original notion 'periodic data at infinity' given by D. Fiebig and U.-R. Fiebig in [3]. Instead of existence of a sequence of periodic points $x^{(n)} \in \operatorname{Per}_k(\sigma) := \big\{x \in X | \sigma^k(x) = x\big\}$ of period k, we request

$$x^{(n)} \in \operatorname{Per}_k^0(\sigma) := \big\{x \in \operatorname{Per}_k(\sigma) | \forall 0 < j < k : \sigma^j(x) \neq x\big\}$$

to have least period k. This difference yields the possibility to prove certain restrictions concerning the canonical-boundary representation, which are not obvious from the original definition (see one of the examples below).

Proposition 3.10: *An arbitrary point in the canonical boundary $(Z(X), \Theta_X)$ cannot be mapped to another boundary point by the extension of some automorphism of the transitive, locally compact countable state Markov shift (X, σ), if their periodic data at infinity do not coincide*

$$\forall z_1, z_2 \in Z(X) : (\exists \varphi \in \operatorname{Aut}(\sigma) : \hat{\varphi}(z_1) = z_2) \Rightarrow \Omega(z_1) = \Omega(z_2).$$

Proof: Suppose there is an automorphism $\varphi \in \operatorname{Aut}(\sigma)$ with $\hat{\varphi}(z_1) = z_2$. For any $k \in \Omega(z_1)$ choose a sequence $(x^{(n)} \in \operatorname{Per}_k^0(\sigma))_{n \in \mathbb{N}}$ converging to z_1 with respect to $\hat{d}$. Its image $(\varphi(x^{(n)}))_{n \in \mathbb{N}}$ consists of elements in $\operatorname{Per}_k^0(\sigma)$ and due to continuity of $\hat{\varphi}$ converges to $z_2 = \hat{\varphi}(z_1)$, so $k \in \Omega(z_2)$. The converse inclusion is proved using φ^{-1} instead of φ. ∎

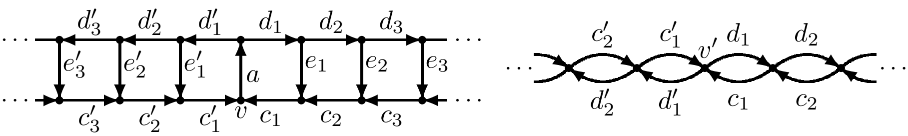

Figure 1. Graph presentations of two transitive, locally compact countable state Markov shifts. Either canonical boundary consists of two Θ_X-orbits of length 2. The automorphism group of the boundary system contains 8 elements.

To round off this section we explicitly determine the images of the canonical-boundary representation for some examples using the results obtained so far:

First have a look at the Markov shifts (X, σ) presented in Figure 1 with

$$\mathrm{Aut}(\Theta_X) \cong \{\mathrm{Id}, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$$

The image of β is pinned down to $\beta(\mathrm{Aut}(\sigma)) \cong \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$ (the Kleinian-four-group) using Theorem 3.6. In both cases it is generated by the shift map σ and the involutoric 1-block-code swapping the left and right halves of each graph ($c_i \longleftrightarrow c'_i$, $d_i \longleftrightarrow d'_i$, $e_i \longleftrightarrow e'_i$). The other four boundary-automorphisms in $\mathrm{Aut}(\Theta_X)$ do not satisfy the congruence relation modulo $\mathrm{per}(X)$ and are thus excluded.

Next glue together one half of each of the two graphs in Figure 1 identifying the vertices $v = \mathfrak{i}(a)$ and $v' = \mathfrak{i}(d_1)$. Once more this yields an edge shift with canonical boundary consisting of two Θ_X-orbits of length 2 and $\mathrm{Aut}(\Theta_X)$ is the same 8-element group as before. But now there is a difference in the periodic data at infinity for the two Θ_X-orbits. The Ω-set for the Θ_X-orbit coming from the left graph in Figure 1 is empty, whereas the two boundary points of the other half can be approximated by sequences of periodic points of least period any positive, even integer (the Ω-set equals $2\mathbb{N}$). Hence following from Proposition 3.10 the extension of any automorphism $\varphi \in \mathrm{Aut}(\sigma)$ has to map each Θ_X-orbit onto itself. Moreover both orbits cannot be rotated by a different amount modulo 2 ($= \mathrm{per}(X)$) according to Corollary 3.7. The inner structure of this example is so rigid that the image of the canonical-boundary representation $\beta(\mathrm{Aut}(\sigma)) = \{\mathrm{Id}_{Z(X)}, \Theta_X\} = \langle \Theta_X \rangle$ is as small as possible, being generated by σ.

Finally we show that our definition of the periodic data at infinity reveals more restrictions than the original one: For this take the left graph in Figure 1 and attach to every vertex $\mathfrak{t}(d_n)$ one simple loop of length 2 and at every vertex $\mathfrak{t}(d'_n)$ two such loops ($n \in \mathbb{N}$). In the original definition of the periodic data at ∞ the two Θ_X-orbits cannot be distinguished; both Ω-sets equal $2\mathbb{N}$. With our definition however the Ω-set for the left Θ_X-orbit is still $2\mathbb{N}$ but that of the Θ_X-orbit on the right contains just one element — the number 2. Therefore the image of the canonical-boundary representation is again minimal.

As the canonical boundary is the same for all these examples the canonical-boundary representation yields additional information about the Markov shift and is a finer conjugacy invariant.

4. Comparability of the path-structure at ∞ – a new conjugacy invariant

We consider the set $\mathrm{Orb}(Z(X)) := {}^{Z(X)}/_{\langle\Theta_X\rangle}$ of Θ_X-orbits of the canonical boundary $(Z(X), \Theta_X)$ of a transitive, locally compact countable state Markov shift (X, σ). If $Z(X)$ is constructed as in Section 2 via an ascending sequence $(K_n)_{n\in\mathbb{N}}$ of finite edge-subsets $K_n \subsetneq E$ of some graph presentation $G = (V, E)$, the quotient map π: $Z(X) \to \mathrm{Orb}(Z(X))$ is a formal projection

$$\pi((H_n, i_n)_{n\in\mathbb{N}}) := (H_n)_{n\in\mathbb{N}}$$

that drops the additional information coming from the periodic labeling. Like the canonical boundary $Z(X)$ the set of Θ_X-orbits $\mathrm{Orb}(Z(X))$ does not depend on the choice of the sequence $(K_n)_{n\in\mathbb{N}}$.

A point $x \in X$ is called forward-asymptotic to some Θ_X-orbit $\mathcal{O} \in \mathrm{Orb}(Z(X))$, if the distance between $\mathcal{O}$ and the σ-forward orbit of x tends to zero with respect to the quotient-metric, i.e. $\inf\{\hat{d}(\sigma^n(x), z) | z \in \mathcal{O}\} \overset{n\to\infty}{\longrightarrow} 0$. Here $\hat{d}$ denotes the canonical metric as stated at the end of Section 2. x is called backward-asymptotic to $\mathcal{O}$, if $\inf\{\hat{d}(\sigma^{-n}(x), z) | z \in \mathcal{O}\} \overset{n\to\infty}{\longrightarrow} 0$.

Now we define a binary relation on the Θ_X-orbits of the canonical boundary. This relation reflects the path-structure of some graph presentation near each of the Θ_X-orbits:

Definition 4.1: Let G_1, G_2 be two locally finite, strongly connected directed graphs with countably infinite edge-sets. Let the canonical boundaries $(Z(X_1), \Theta_{X_1})$, $(Z(X_2), \Theta_{X_2})$ of the corresponding edge shifts X_1, X_2 be generated by ascending sequences $(K_n^{(1)})_{n\in\mathbb{N}}$, $(K_n^{(2)})_{n\in\mathbb{N}}$ of finite edge-subsets.

Some Θ_{X_1}-orbit $\mathcal{O}_1 = (H_n^{(1)})_{n\in\mathbb{N}} \in \mathrm{Orb}(Z(X_1))$ is called **embeddable into** some Θ_{X_2}-orbit $\mathcal{O}_2 = (H_n^{(2)})_{n\in\mathbb{N}} \in \mathrm{Orb}(Z(X_2))$ with respect to its path-structure at infinity, if and only if:

For all points $x, y \in X_1$ with x forward-asymptotic to $\mathcal{O}_1$ and y backward-asymptotic to $\mathcal{O}_1$ and for every $N \in \mathbb{N}$ there exist natural numbers $n_1, n_2 \geq N$ and indices $i, j, k, l \in \mathbb{N}$ satisfying $i \leq j$, $-k \leq -l$ such that the infinite rays $x_{[i,\infty)} = x_i \cdots x_j \cdots$ and $y_{(-\infty,-l]} = \cdots y_{-k} \cdots y_{-l}$ run totally in $H_{n_1}^{(1)}$ and additionally there are edges $e, f \in E_{H_{n_2}^{(2)}}$ that fulfill the following inequality for all path-lengths $m \in \mathbb{N}$:

$$\begin{aligned}\#\{b_1 \cdots b_m | x_{[i,j]} b_1 \cdots b_m y_{[-k,-l]} \in \mathcal{B}_{\tilde{m}+2}(X_1) \wedge \forall 1 \leq \mu \leq m : b_\mu \in E_{H_{n_1}^{(1)}}\} \\ \leq \#\{b_1 \cdots b_{\tilde{m}} | e b_1 \cdots b_{\tilde{m}} f \in \mathcal{B}_{\tilde{m}+2}(X_2) \wedge \forall 1 \leq \mu \leq \tilde{m} : b_\mu \in E_{H_{n_2}^{(2)}}\}\end{aligned}$$

where $\tilde{m} := m + (j - i) + (k - l)$ and $\mathcal{B}_{\tilde{m}+2}(X_1)$, $\mathcal{B}_{\tilde{m}+2}(X_2)$ denotes the set of admissible words of length $\tilde{m} + 2$ in X_1 respectively X_2.
Two orbits $\mathcal{O}_1 = (H_n^{(1)})_{n\in\mathbb{N}} \in \mathrm{Orb}(Z(X_1))$ and $\mathcal{O}_2 = (H_n^{(2)})_{n\in\mathbb{N}} \in \mathrm{Orb}(Z(X_2))$ are called **comparable with respect to their path-structure at** ∞, if both $\mathcal{O}_1$ is embeddable into $\mathcal{O}_2$ and $\mathcal{O}_2$ is embeddable into $\mathcal{O}_1$.

Since a priori the definition of the path-structure at infinity relies heavily on the graph presentation, we have to show that any topological conjugacy between two transitive, locally compact countable state Markov shifts respects this reflexive, symmetric relation. Hence this relation becomes meaningful, even for non-graph presentations and yields a new conjugacy invariant that forces the following condition necessary for the existence of a topological conjugacy:

THEOREM 4.2: *Whenever two transitive, locally compact countable state Markov shifts $(X_1, \sigma_1), (X_2, \sigma_2)$ are topologically conjugate, there has to be a bijective map ω: $\mathrm{Orb}(Z(X_1)) \to \mathrm{Orb}(Z(X_2))$ between the orbits of their canonical boundaries $(Z(X_1), \Theta_{X_1})$ and $(Z(X_2), \Theta_{X_2})$ such that — in any graph presentation G_1 respectively G_2 — every Θ_{X_1}-orbit $\mathcal{O}_1 \in \mathrm{Orb}(Z(X_1))$ is comparable with respect to its path-structure at ∞ to its image $\mathcal{O}_2 := \omega(\mathcal{O}_1) \in \mathrm{Orb}(Z(X_2))$.*

Proof: Assume there exists a topological conjugacy γ: $X_1 \to X_2$, then its unique extension $\hat{\gamma}$: $\hat{X}_1 \to \hat{X}_2$ onto the canonical compactifications induces a map from the set of Θ_{X_1}-orbits to the set of Θ_{X_2}-orbits which is bijective and respects the path-structure relation as desired:

First observe that the map $\omega := \pi_{X_2} \circ \hat{\gamma} \circ \pi_{X_1}^{-1}$: $\mathrm{Orb}(Z(X_1)) \to \mathrm{Orb}(Z(X_2))$ is well-defined, as $\hat{\gamma}$ commutes with the extended shift maps $\hat{\sigma}_1, \hat{\sigma}_2$.

As usual, let the canonical boundaries $(Z(X_1),\Theta_{X_1}),(Z(X_2),\Theta_{X_2})$ be constructed via ascending sequences $(K_n^{(1)})_{n\in\mathbb{N}},(K_n^{(2)})_{n\in\mathbb{N}}$ of edge-sets of strongly connected, finite subgraphs in G_1 respectively G_2, where $K_1^{(1)}$ already comprises the edges of a shortest loop p in G_1. Let $\mathcal{O}_1=(H_n^{(1)})_{n\in\mathbb{N}}\in\mathrm{Orb}(Z(X_1))$ be an arbitrary Θ_{X_1}-orbit and let $\mathcal{O}_2=(H_n^{(2)})_{n\in\mathbb{N}}:=\omega(\mathcal{O}_1)\in\mathrm{Orb}(Z(X_2))$ be its image.

Define two points $u:=p^\infty.u_0u_1u_2\cdots\in X_1$ being forward-asymptotic to $\mathcal{O}_1$ and $w:=\cdots w_{-3}w_{-2}w_{-1}.p^\infty\in X_1$ being backward-asymptotic to $\mathcal{O}_1$. Observe that any right-(left-)infinite ray forward-(backward-)asymptotic to $\mathcal{O}_1$ can be realized as a subray in $u_{[0,\infty)}$ respectively $w_{(-\infty,-1]}$. $\gamma(p^\infty)=q^\infty\in\mathrm{Per}^0_{|p|}(\sigma_2)$ yields a finite block $q\in\mathcal{B}_{|p|}(X_2)$ that has a finite coding-length $L\in|p|\,\mathbb{N}$. Thus the images of u and w under γ look like $U=q^\infty U_{-L}\cdots U_{-1}.U_0U_1\cdots\in X_2$ and $W=\cdots W_{-2}W_{-1}.W_0\cdots W_{L-1}q^\infty\in X_2$. By construction of u one has

$$\inf\{\hat{d}_{X_1}(\sigma_1{}^n(u),z)|z\in\mathcal{O}_1\}\stackrel{n\to\infty}{\longrightarrow}0.$$

Continuity of $\hat{\gamma}$ with respect to $\hat{d}_{X_1}$ and $\hat{d}_{X_2}$ forces

$$\inf\{\hat{d}_{X_2}(\sigma_2{}^n(\gamma(u)),z)|z\in\mathcal{O}_2\}\stackrel{n\to\infty}{\longrightarrow}0$$

i.e., $U=\gamma(u)$ is forward-asymptotic to $\mathcal{O}_2$. For every $n\in\mathbb{N}$ there exists some index $i\in\mathbb{N}$ such that the right-infinite ray $U_{[i,\infty)}$ is contained in $H_n^{(2)}$. Likewise $W=\gamma(w)$ is backward-asymptotic to $\mathcal{O}_2$ and for every $n\in\mathbb{N}$ there is an index $i\in\mathbb{N}$ with $W_{(-\infty,-i]}$ completely running in $H_n^{(2)}$.

The preimage of the compact set $\bigcup_{k\in K_{n_2}^{(2)}}{}_0[k]\subsetneq X_2$ under γ is contained in a finite union of zero-cylinders in X_1 for any $n_2\in\mathbb{N}$; so there exists $n_1\in\mathbb{N}$ such that

$$\text{(KH)}\qquad \begin{aligned}&\gamma^{-1}\Big(\bigcup_{k\in K_{n_2}^{(2)}}{}_0[k]\Big)\subseteq\bigcup_{k\in K_{n_1}^{(1)}}{}_0[k]\quad\text{and thus,}\\ &\gamma^{-1}\Big(\bigcup_{k\in K_{n_2}^{(2)}}{}_0[k]\Big)\cap\bigcup_{e\in E_{H_{n_1}^{(1)}}}{}_0[e]=\emptyset.\end{aligned}$$

Fix $n_2\in\mathbb{N}$ large enough and let $i\in\mathbb{N}$ be the minimal index with $U_l,W_{-l}\in E_{H_{n_2}^{(2)}}$ for all $l\geq i$. Due to the finite coding-lengths of the rays $u_{(-\infty,.]}$ and $w_{[.,\infty)}$ containing only finitely many distinct symbols, it is possible to find $j\in\mathbb{N}$, $j>i$ satisfying $u_l,w_{-l}\in E_{H_{n_1}^{(1)}}$ for all $l\geq j$ where $n_1\in\mathbb{N},\ n_1\geq n_2$ is chosen apt to n_2 according to (KH) and

$$\begin{aligned}\gamma((p^\infty u_0\cdots u_i\cdots u_{j-1}u_j]_{j-i})&\subseteq(q^\infty U_{-L}\cdots U_0\cdots U_{i-1}U_i]_0\\ \gamma({}_{i-j}[w_{-j}w_{-j+1}\cdots w_{-i}\cdots w_{-1}p^\infty))&\subseteq{}_0[W_{-i}W_{-i+1}\cdots W_0\cdots W_{L-1}q^\infty).\end{aligned}$$

Finally there is $k \in \mathbb{N},\ k \geq j$ such that

$$\gamma((p^\infty u_0 \cdots u_i \cdots u_{k-1} u_k]_{k-i}) \subseteq (q^\infty U_{-L} \cdots U_i \cdots U_{j-1} U_j]_{j-i}$$
$$\gamma({}_{i-k}[w_{-k} w_{-k+1} \cdots w_{-i} \cdots w_{-1} p^\infty)) \subseteq {}_{i-j}[W_{-j} W_{-j+1} \cdots W_{-i} \cdots W_{L-1} q^\infty).$$

Now compare the path-structure between the vertices $\mathfrak{t}(u_k)$ and $\mathfrak{i}(w_{-k})$ within the subgraph $H^{(1)}_{n_1}$ and the path-structure between $\mathfrak{t}(U_j)$ and $\mathfrak{i}(W_{-j})$ within $H^{(2)}_{n_2}$:

Let $B_m := \{b_1 \cdots b_m | u_k b_1 \cdots b_m w_{-k} \in \mathcal{B}_{m+2}(X_1) \wedge \forall 1 \leq \mu \leq m : b_\mu \in E_{H^{(1)}_{n_1}}\}$ ($m \in \mathbb{N}$) be the set of all paths of length m connecting $\mathfrak{t}(u_k)$ with $\mathfrak{i}(w_{-k})$ that are contained in $H^{(1)}_{n_1}$. To every block $b = b_1 \cdots b_m \in B_m$ construct some point $x_b := p^\infty u_0 \cdots u_{i-1}.u_i \cdots u_k b_1 \cdots b_m w_{-k} \cdots w_{-1} p^\infty \in X_1$. Its image under γ looks like $\gamma(x_b) = q^\infty U_{-L} \cdots U_{i-1}.U_i \cdots U_j \underbrace{?\,?\,?}\, W_{-j} \cdots W_{L-1} q^\infty \in X_2$ where $U_j, W_{-j} \in E_{H^{(2)}_{n_2}}$ is guaranteed by the choice of i.

Since $u_{j+1} \cdots u_k b_1 \cdots b_m w_{-k} \cdots w_{-j-1}$ contains only edges from $E_{H^{(1)}_{n_1}}$, according to (KH) no edge of the unknown block can be an element of $K^{(2)}_{n_2}$. Hence the whole unknown block does consist of edges from $E_{H^{(2)}_{n_2}}$ and the number of paths of length $m+2(k-j)$ within $H^{(2)}_{n_2}$ connecting $\mathfrak{t}(U_j)$ with $\mathfrak{i}(W_{-j})$ has to be larger or equal to the number of paths $u_{j+1} \cdots u_k b_1 \cdots b_m w_{-k} \cdots w_{-j-1}$ ($b_1 \cdots b_m \in B_m$) within $H^{(1)}_{n_1}$ for any $m \in \mathbb{N}$. This shows $\mathcal{O}_1$ being embeddable into $\mathcal{O}_2$ with respect to the path-structure at ∞. The same argument for γ^{-1} instead of γ proves the embeddability of $\mathcal{O}_2$ into $\mathcal{O}_1$, so $\mathcal{O}_1$ and $\mathcal{O}_2 = \omega(\mathcal{O}_1)$ are in fact comparable. ∎

Reformulating Theorem 4.2 in terms of automorphisms yields the following statement about the image of β:

Corollary 4.3: *Denote by $(Z(X), \Theta_X)$ the canonical boundary of some transitive, locally compact countable state Markov shift (X, σ). A Θ_X-orbit $\mathcal{O}_1 \in \mathrm{Orb}(Z(X))$ cannot be mapped onto $\mathcal{O}_2 \in \mathrm{Orb}(Z(X))$ via the extension $\hat{\varphi}$ of some automorphism $\varphi \in \mathrm{Aut}(\sigma)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are not comparable with respect to their path-structure at infinity.*

The image of the canonical-boundary representation contains only boundary-automorphisms on $(Z(X), \Theta_X)$ that respect the path-structure relation, mapping all Θ_X-orbits to comparable ones.

Proof: The result immediately follows from Theorem 4.2; just put

$$(X_1, \sigma_1) = (X_2, \sigma_2) := (X, \sigma), \quad \gamma := \varphi,$$

$$G_1 = G_2 := G \text{ some graph presentation}$$

and $(K_n^{(1)})_{n\in\mathbb{N}} = (K_n^{(2)})_{n\in\mathbb{N}}$. ∎

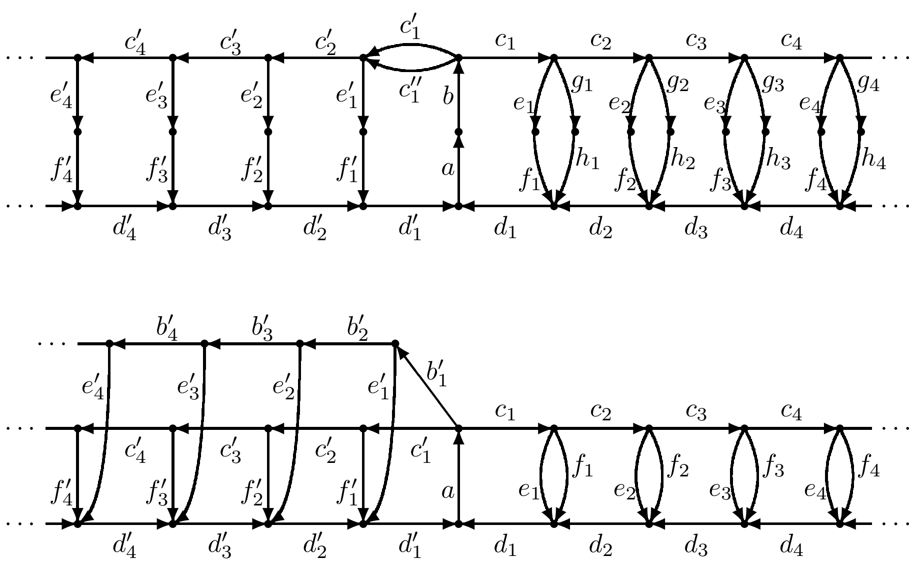

Figure 2. Pictured are the graph presentations of two locally compact countable state Markov shifts with very limited inner symmetry. The image of the canonical-boundary representation is thus fairly small within the group of all boundary-automorphisms.

The canonical boundaries for the two examples of transitive, locally compact countable state Markov shifts (X, σ) presented in Figure 2 each consist of two Θ_X-orbits of length 2. As before we may apply Theorem 3.6 to get

$$\beta(\mathrm{Aut}(\sigma)) \leq \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}\,.$$

Since the periodic data at ∞ of both Θ_X-orbits are equal, Proposition 3.10 cannot be used to further restrict the image of β. However looking at the path-structure at ∞ finally settles the size of the image: We recognize the two Θ_X-orbits as non-comparable to each other; one could say the right half of the graph is not embeddable into the left half. So following from Corollary 4.3 the image of the canonical-boundary representation is minimal $\beta(\mathrm{Aut}(\sigma)) = \beta(\langle\sigma\rangle) = \{\mathrm{Id}, \Theta_X\}$ for both examples.

Cutting in half each of the example graphs in Figure 2 along the upward directed edge(s) a, b respectively a (duplicating these, such that both halves contain one copy) yields two pairs of edge shifts — each pair with equal zeta function, equal canonical boundary and equal periodic data at ∞. But still Theorem 4.2 tells the two subshifts in each pair apart and disproves the existence of a topological conjugacy between them. This makes the relation 'comparability of the path-structure at ∞' a useful supplement to the set of conjugacy invariants.

We remark that removing the edges e_1, f_1, e_1' and f_1' from the bottom graph in Figure 2 yields a new edge shift, which is topologically conjugate to the Markov shift corresponding to the top graph in Figure 2, i.e. the boundary-orbits of the right respectively left halves of both edge shifts are comparable with respect to their path-structure at infinity.

As another consequence following from Corollary 4.3. In general, not every permutation of some infinite set of periodic points of an arbitrary fixed period, which is compatible with the shift and which induces a well defined action on the boundary points, can be extended to an automorphism of the whole Markov shift.

We denote by $\mathrm{Aut}(\mathrm{Per}_n^0(\sigma), \sigma)$ the group of all permutations τ on $\mathrm{Per}_n^0(\sigma)$ which respect the σ-orbit structure, i.e. $\tau \circ \sigma = \sigma \circ \tau$, as well as the following obvious necessary conditions: Whenever a sequence $(p_i \in \mathrm{Per}_n^0(\sigma))_{i\in\mathbb{N}}$ converges to some boundary point its image sequence $(\tau(p_i))_{i\in\mathbb{N}}$ has to converge to some boundary point and whenever a sequence $(p_i \in \mathrm{Per}_n^0(\sigma))_{i\in\mathbb{N}}$ eventually leaves every compact set in (X, σ), its image $(\tau(p_i))_{i\in\mathbb{N}}$ has to leave eventually every compact subset.

Now we can ask whether every element of $\mathrm{Aut}(\mathrm{Per}_n^0(\sigma), \sigma)$ gives rise to an automorphism of (X, σ). In Example 4.4 below we disprove this for all sets $\mathrm{Per}_n^0(\sigma)$ $(n \in \mathbb{N})$ for a slight modification of one of the edge shifts from Figure 2.

Example 4.4: The LIFT-hypothesis for subsets of periodic points is not valid for transitive, locally compact countable state Markov shifts independent of the size of the period.

Take the top graph from Figure 2 and add to every vertex $\mathfrak{i}(d_i)$ and $\mathfrak{i}(f_i')$ two self-loops k_i, l_i or k_i', l_i' respectively ($i \in \mathbb{N}$). This produces a new edge shift (X, σ).

Consider some permutation $\tau \in \mathcal{S}_{\mathrm{Per}_n^0(\sigma)}$ on the set of periodic points $\mathrm{Per}_n^0(\sigma)$ of least period $n \in \mathbb{N}$, which exchanges the two subsets $\{(k_i {l_i}^{n-1})^\infty | i \in \mathbb{N}\}$ and $\{(k_i' {l_i'}^{n-1})^\infty | i \in \mathbb{N}\}$:

$$\forall i \in \mathbb{N} : \tau((k_i {l_i}^{n-1})^\infty) := (k_i' {l_i'}^{n-1})^\infty,\ \tau((k_i' {l_i'}^{n-1})^\infty) := (k_i {l_i}^{n-1})^\infty$$

and assume $\sigma \circ \tau = \tau \circ \sigma$ (τ may act at will on the remaining σ-orbits in $\mathrm{Per}_n^0(\sigma)$). Obviously $\tau \in \mathrm{Aut}(\mathrm{Per}_n^0(\sigma), \sigma)$. We claim that such a permutation τ cannot occur as the restriction of any automorphism $\varphi \in \mathrm{Aut}(\sigma)$:

Denote by $z_1 \in Z(X)$ the right and by $z_2 \in Z(X)$ the left boundary point. (Note that now $(Z(X), \Theta_X)$ consists of two Θ_X-orbits, each of length 1.) Due to the convergence $\hat{d}((k_i {l_i}^{n-1})^\infty, z_1) \overset{i\to\infty}{\longrightarrow} 0$ and $\hat{d}((k_i' {l_i'}^{n-1})^\infty, z_2) \overset{i\to\infty}{\longrightarrow} 0$, to realize $\varphi|_{\mathrm{Per}_n^0(\sigma)} = \tau$ would force the existence of some $\varphi \in \mathrm{Aut}(\sigma)$ such that $\hat{\varphi}(z_1) = z_2$. So it suffices to prove $\mathcal{O}_1 := \{z_1\}$ being non-comparable to $\mathcal{O}_2 := \{z_2\}$ concerning the path-structure at ∞.

Using the notation from Definition 4.1, for every ray $x_{[i,\infty)}$ and $y_{(-\infty,-l]}$, both asymptotic to $\mathcal{O}_1$ and completely contained in $H_{n_1}^{(1)}$, for any edges $e, f \in E_{H_{n_2}^{(2)}}$ and every $\tilde{m} = m + (j-i) + (k-l)$ we define:

$$B_{e,f}(\tilde{m}) := \{b_1 \cdots b_{\tilde{m}} | e b_1 \cdots b_{\tilde{m}} f \in \mathcal{B}_{\tilde{m}+2}(X) \wedge \forall 1 \le \mu \le \tilde{m} : b_\mu \in E_{H_{n_2}^{(2)}}\}$$

and

$$\begin{aligned} B_{x_{[i,j]}, y_{[-k,-l]}}(m) := \{b_1 \cdots b_m | x_{[i,j]} b_1 \dots b_m y_{[-k,-l]} \in \mathcal{B}_{\tilde{m}+2}(X) \wedge \\ \forall 1 \le \mu \le m : b_\mu \in E_{H_{n_1}^{(1)}}\}. \end{aligned}$$

On the one hand $\# B_{e,f}(\tilde{m}) \le 2^{\tilde{m}}$ independently of e, f; on the other hand for large path-lengths $m \in \mathbb{N}$ there is some constant $I \in \mathbb{N}$ depending only on the choice of x_j and y_{-k} (I is the minimal distance from $\mathfrak{t}(x_j)$ to $\mathfrak{i}(y_{-k})$ minus 1) satisfying:

$$\begin{aligned} &\# B_{x_{[i,j]}, y_{[-k,-l]}}(m) \\ &\quad = 2^{m-I} + 2^{m-I-2}(m-I-2) + \underbrace{2^{m-I-5}(m-I-4)(m-I-3) + \cdots}_{\ge 0} \end{aligned}$$

For $m > 2^{(j-i)+(k-l)+I+2} + I - 2$ this implies $\# B_{e,f}(\tilde{m}) < \# B_{x_{[i,j]}, y_{[-k,-l]}}(m)$; thus $\mathcal{O}_1$ cannot be embedded into $\mathcal{O}_2$. Following from Corollary 4.3 the extension of every automorphism acts on $Z(X)$ like the identity and the image of the canonical-boundary representation is minimal $\beta(\mathrm{Aut}(\sigma)) = \{\mathrm{Id}_{Z(X)}\}$.

We point out that even though the canonical boundary (two identical Θ_X-orbits) and the periodic data at ∞ (identical Ω-sets for both boundary points) do not give any restrictions, the automorphism group $\mathrm{Aut}(\sigma)$ acts on every set $\mathrm{Per}_n^0(\sigma)$ like a proper subgroup of the full automorphism group $\mathrm{Aut}(\mathrm{Per}_n^0(\sigma), \sigma)$.

Finally we give a counterexample to the LIFT-hypothesis for ascending sequences of compact subshifts. Moreover in our example these compact subshifts are SFTs; hence disproving at the same time the specialized LIFT-hypothesis for ascending sequences of SFTs.

Example 4.5: The LIFT-hypothesis for ascending sequences of SFTs is not valid for transitive, locally compact countable state Markov shifts.

Look at the edge shift (X, σ) corresponding to the bottom graph displayed in Figure 2, together with the included, nested sequence of transitive SFTs $((X_n, \sigma_n))_{n\in\mathbb{N}}$ defined via the strongly connected, finite subgraphs $G_n = (V_n, E_n)$ with

$$E_n := \{a, c_i, d_i, e_i, f_i, b_i', c_i', d_i', e_i', f_i' \mid i \leq n\} \quad \text{and} \quad V_n := \{\mathfrak{i}(e) \mid e \in E_n\}$$

On every X_n we have an involutoric marker-automorphism $\varphi_n \in \mathrm{Aut}(\sigma_n)$ swapping left and right; more precisely, φ_n scans a point $x \in X_n$ and exchanges any of the following blocks marked by a:

$$\left.\begin{array}{l} ac_1 \cdots c_i e_i d_i \cdots d_1 a \longleftrightarrow ab_1' \cdots b_i' e_i' d_i' \cdots d_1' a \\ ac_1 \cdots c_i f_i d_i \cdots d_1 a \longleftrightarrow ac_1' \cdots c_i' f_i' d_i' \cdots d_1' a \end{array}\right. \quad \forall i \leq n.$$

If we regard X_n as a subset in X, every φ_n induces an automorphism $\tilde{\varphi}_n \in \mathrm{Aut}(\sigma)$ acting only on some finite subgraph/subset of symbols by exchange of the blocks stated above. Since $X_n \subseteq X_{n+1} \subsetneq X$ and $\tilde{\varphi}_m|_{X_n} = \tilde{\varphi}_n|_{X_n} = \varphi_n$ for $n \leq m$ $(n, m \in \mathbb{N})$, the sequence $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ is projective and we may define some map $\tilde{\varphi}\colon \bigcup_{n\in\mathbb{N}} X_n \to \bigcup_{n\in\mathbb{N}} X_n$ putting $\tilde{\varphi}(x) := \tilde{\varphi}_n(x)$ for $x \in X_n$.

However $\tilde{\varphi}$ is not extendible to an automorphism of the whole transitive, locally compact countable state Markov shift (X, σ): Apparently the union $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X with respect to the usual cylinder-topology and X is dense in its canonical compactification $\hat{X}$. So there is at most one continuous map $\hat{\varphi}\colon \hat{X} \to \hat{X}$ satisfying $\hat{\varphi}|_{\bigcup_{n\in\mathbb{N}} X_n} = \tilde{\varphi}$. It is easy to check that this $\hat{\varphi}$ would exchange the two Θ_X-orbits of the canonical boundary. As we have already shown above, these are not comparable concerning the path-structure at ∞. Hence Corollary 4.3 contradicts the existence of $\varphi := \hat{\varphi}|_X \in \mathrm{Aut}(\sigma)$.

Thus the sequence $(\tilde{\varphi}_n \in \mathrm{Aut}(\sigma))_{n\in\mathbb{N}}$ with $\tilde{\varphi}_n(X_n) = X_n$ and $\tilde{\varphi}_m|_{X_n} = \tilde{\varphi}_n|_{X_n}$ for all $m \geq n$ cannot be extended to some $\varphi \in \mathrm{Aut}(\sigma)$ such that $\varphi|_{X_n} = \tilde{\varphi}_n|_{X_n}$ for $n \in \mathbb{N}$ and the LIFT-hypothesis does not hold for the SFT-sequence $((X_n, \sigma_n))_{n\in\mathbb{N}}$.

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